ON A SUBCLASS OF 5-DIMENSIONAL SOLVABLE LIE ALGEBRAS WHICH HAVE 3-DIMENSIONAL COMMUTATIVE DERIVED IDEAL

Le Anh Vu
Department of Mathematics and Informatics
Ho Chi Minh City University of Pedagogy
e-mail:leanhvu@hcmup.edu.vn

Abstract

The paper presents a subclass of the class of MD5-algebras and MD5-groups, i.e., five dimensional solvable Lie algebras and Lie groups such that their orbits in the co-adjoint representation (K-orbit) are orbit of zero or maximal dimension. The main results of the paper is the classification up to an isomorphism of all MD5-algebras $\mathcal{G}$ with the derived ideal $\mathcal{G}^1 := [\mathcal{G}, \mathcal{G}]$ is a 3-dimensional commutative Lie algebra.

[0]**Key words**: Lie group, Lie algebra, MD5-group, MD5-algebra, K-orbits.
2000AMS Mathematics Subject Classification: Primary 22E45, Secondary 46E25, 20C20.

Introduction

In 1962, studying theory of representations, Kirillov [3] introduced the Orbit Method. This method quickly became the most important method in the theory of representations of Lie groups. The Kirillov's Orbit Method immediately was expanded by Kostant, Auslander, Do Ngoc Diep, etc. Using the Kirillov's Orbit Method, we can obtain all the unitary irreducible representations of solvable and simply connected Lie groups. The importance of Kirillov's Orbits Method is the co-adjoint representation (K-representation). Therefore, it is meaningful to study the K-representation in the theory of representations of Lie groups.

The structure of solvable Lie groups and Lie algebras is not to complicated, although the complete classification of them is unresolved up to now. In 1980, studying the Kirillov's Orbit Method, D. N. Diep [2] introduced the class of Lie groups and Lie algebras MD. Let G be an n-dimensional real Lie group. It is called an MDn-group iff its orbits in the K-representation (i.e. K-orbits) are orbits of dimension zero or maximal dimension. The corresponding Lie algebra of G is called an MDn-algebra. Thus, classification and study of K-representation of the class of MDn-groups and MDn-algebras is the problem of great interest. Because of all Lie algebras of n dimension (with $n \leq 3$) were listed easily, we have to consider MDn-groups and MDn-algebras with $n \geq 4$.

In 1984, Dao Van Tra ([5]) was listed all MD4-algebras. In 1992, all MD4-algebras were classified up to an isomorphism by the author (see [6], [7], [8]). Until now, no complete classification of MDn-algebras with $n \geq 5$ is known. Three first examples of MD5-algebras and MD5-groups can be found in [9] and some different MD5-algebras and MD5-groups can be found in [10]. In this paper we shall give the classification up to an isomorphism of all MD5-algebras $\mathcal{G}$ with the derived ideal $\mathcal{G}^1 := [\mathcal{G}, \mathcal{G}]$ is a 3-dimensional commutative Lie algebra. The complete classification of all MD5-algebras

will be presented in the next paper.

1 Preliminaries

At first, we recall in this section some preliminary results and notations which will be used later. For details we refer the reader to References [2], [3], [4].

1.1 Lie Groups and Lie Algebras

Definition 1.1. *A real Lie group of dimension n is a C^∞-manifold G endowed with a group structure such that the map $(g, h) \mapsto g.h^{-1}$ from $G \times G$ into G is C^∞-differentiable.*

Definition 1.2. *A real Lie algebra $\mathcal{G}$ of dimension n is an n-dimensional real vector space together with a skew-symmetric bilinear map (X, Y) $\mapsto$ [X, Y] from $\mathcal{G} \times \mathcal{G}$ into $\mathcal{G}$ (which is called the Lie bracket) such that the following Jacobi identity is satisfied : [[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0 for every X, Y, Z $\in \mathcal{G}$.*

1.2 The co-adjoint Representation, K-orbits MDn-Groups and MDn-Algebras

Each Lie group G determines a Lie algebra Lie(G) = $\mathcal{G}$ as the tangent space T_eG of G at the identity with the Lie bracket is defined by the commutator. Inversely, each real Lie algebra $\mathcal{G}$ is associated to one connected and simply connected Lie group G such Lie(G) = $\mathcal{G}$. For each g $\in$ G, we denote the internal automorphism associated to g by $A_{(g)}$. So $A_{(g)} : G \longrightarrow G$ is defined as follows

$$A_{(g)}(x) := g.x.g^{-1}, \forall x \in G.$$

This automorphism induces the following map

$$A_{(g)_*} : \mathcal{G} \longrightarrow \mathcal{G}$$

$$X \longmapsto A_{(g)_*}(X) := \frac{d}{dt}[g.exp(tX)g^{-1}] \mid_{t=0}$$

which is called the tangent map of $A_{(g)}$.

Definition 1.3. *The action*

$$Ad : G \longrightarrow Aut(\mathcal{G})$$

$$g \longmapsto Ad(g) := A_{(g)_*}$$

is called the adjoint representation of G in $\mathcal{G}$.

Definition 1.4. *The action*

$$K : G \longrightarrow Aut(\mathcal{G}^*)$$

$$g \longmapsto K_{(g)}$$

such that

$$\langle K_{(g)}F, X\rangle := \langle F, Ad(g^{-1})X\rangle; \quad (F \in \mathcal{G}^*,\ X \in \mathcal{G})$$

is called the co-adjoint representation of G in $\mathcal{G}^*$.

Definition 1.5. *Each orbit of the co-adjoint representation of G is called a K-orbit of G.*

Thus, for every $F \in \mathcal{G}^*$, the K-orbit containing F is defined as follows

$$\Omega_F := \{K_{(g)}F/g \in G\}.$$

The dimension of every K-orbit of G is always even. In order to define the dimension of the K-orbits Ω_F, it is useful to consider the skew-symmetric bilinear form B_F on $\mathcal{G}$ as follows

$$B_F(X, Y) := \langle F, [X, Y]\rangle;\ \forall\, X, Y \in \mathcal{G}.$$

Denote the stabilizer of F under the co-adjoint representation of G in $\mathcal{G}^*$ by G_F and $\mathcal{G}_F := \mathrm{Lie}(G_F)$. We shall need in the sequel the following assertion.

Proposition 1.6 (see [3]). *$KerB_F = \mathcal{G}_F$ and $dim\Omega_F = dim\mathcal{G} - dim\mathcal{G}_F$.* $\square$

Definition 1.7 (see [2]). *An MDn-group is an n-dimensional real solvable Lie group such that its K-orbits are orbits of dimension zero or maximal dimension. The Lie algebra of an MDn-group is called an MDn-algebra.*

The following proposition give a necessary condition in order that a Lie algebra belongs to the class of all MD-algebras.

Proposition 1.8 (see [4]). *Let $\mathcal{G}$ be an MD-algebra. Then its second derived ideal $\mathcal{G}^2 := [[\mathcal{G},\mathcal{G}],[\mathcal{G},\mathcal{G}]]$ is commutative.* $\square$

Note, however, that the converse of this statement in general is not hold. In other words, the above necessary condition is not sufficient one.

2 The Main Result

From now on, $\mathcal{G}$ will denote an Lie algebra of dimension 5. We always choose a suitable basis $(X_1, X_2, X_3, X_4, X_5)$ in $\mathcal{G}$. Then $\mathcal{G}$ isomorphic to $\mathbf{R}^5$ as a real vector space. The notation $\mathcal{G}^*$ will mean the dual space of $\mathcal{G}$. Clearly $\mathcal{G}^*$ can be identified with $\mathbf{R}^5$ by fixing in it the basis $(X_1^*, X_2^*, X_3^*, X_4^*, X_5^*)$ dual to the basis $(X_1, X_2, X_3, X_4, X_5)$.

Theorem 2.1. *Let $\mathcal{G}$ be an MD5-algebra with $\mathcal{G}^1 := [\mathcal{G},\mathcal{G}] \cong \mathbf{R^3}$ (the 3-dimensional commutative Lie algebra).*

I. *Assume that $\mathcal{G}$ is decomposable. Then $\mathcal{G} \cong \mathcal{H} \oplus \mathbf{R}$, where $\mathcal{H}$ is an MD4-algebra.*

II. *Assume that $\mathcal{G}$ is indecomposable. Then we can choose a suitable basis $(X_1, X_2, X_3, X_4, X_5)$ of $\mathcal{G}$ such that $\mathcal{G}^1 = \mathbf{R}.X_3 \oplus \mathbf{R}.X_4 \oplus \mathbf{R}.X_5 \equiv \mathbf{R^3}$, $ad_{X_1} = 0$, $ad_{X_2} \in End(\mathcal{G}^1) \equiv Mat_3(\mathbf{R})$; $[X_1, X_2] = X_3$ and $\mathcal{G}$ is isomorphic to one and only one of the following Lie algebras:*

1. $\mathcal{G}_{5,3,1(\lambda_1,\lambda_2)}$:

$$ad_{X_2} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \lambda_1, \lambda_2 \in \mathbf{R} \setminus \{1\},\ \lambda_1 \neq \lambda_2 \neq 0.$$

2. $\mathcal{G}_{5,3,2(\lambda)}$:

$$ad_{X_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}; \quad \lambda \in \mathbf{R} \setminus \{0, 1\}.$$

3. $\mathcal{G}_{5,3,3(\lambda)}$:

$$ad_{X_2} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \lambda \in \mathbf{R} \setminus \{1\}.$$

4. $\mathcal{G}_{5,3,4}$:

$$ad_{X_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

5. $\mathcal{G}_{5,3,5(\lambda)}$:

$$ad_{X_2} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}; \quad \lambda \in \mathbf{R} \setminus \{1\}.$$

6. $\mathcal{G}_{5,3,6(\lambda)}$:

$$ad_{X_2} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}; \quad \lambda \in \mathbf{R} \setminus \{0, 1\}.$$

7. $\mathcal{G}_{5,3,7}$:

$$ad_{X_2} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

8. $\mathcal{G}_{5,3,8(\lambda,\varphi)}$:

$$ad_{X_2} = \begin{pmatrix} cos\varphi & -sin\varphi & 0 \\ sin\varphi & cos\varphi & 0 \\ 0 & 0 & \lambda \end{pmatrix}; \quad \lambda \in \mathbf{R} \setminus \{0\}, \varphi \in (0, \pi).$$

In order to prove Theorem 2.1 we need some lemmas.

Lemma 2.2. *Under the above notation. We have* $ad_{X_1} \circ ad_{X_2} = ad_{X_2} \circ ad_{X_1}$.

Proof. Using the Jacobi identity for X_1, X_2 and $X_i (i = 3, 4, 5\ )$, we have

$$\begin{aligned} & [[X_1, X_2], X_i] + [[X_2, X_i], X_1] + [[X_i, X_1], X_2] = 0 \\ \Leftrightarrow \quad & [X_1, [X_2, X_i]] - [X_2, [X_1, X_i]] = 0 \\ \Leftrightarrow \quad & ad_{X_1} \circ ad_{X_2}(X_i) = ad_{X_2} \circ ad_{X_1}(X_i);\ i = 3, 4, 5 \\ \Leftrightarrow \quad & ad_{X_1} \circ ad_{X_2} = ad_{X_2} \circ ad_{X_1}. \end{aligned}$$

□

Lemma 2.3 (see[2], [4]). *If* $\mathcal{G}$ *is an MD-algebra and* $F \in \mathcal{G}^*$ *is not perfectly vanishing on* $\mathcal{G}^1$, *i.e. there exists* $U \in \mathcal{G}^1$ *such that* $\langle F, U \rangle \neq 0$, *then the K-orbit* Ω_F *is the one of maximal dimension.*

Proof. Assume that Ω_F is not a K-orbit of maximal dimension, i.e. $dim\Omega_F = 0$. This means that

$$dim\mathcal{G}_F = dim\mathcal{G} - dim\Omega_F = dim\mathcal{G}.$$

So $Ker B_F = \mathcal{G}_F = \mathcal{G} \supset \mathcal{G}^1$ and F is perfectly vanishing on $\mathcal{G}^1$. This contradicts the supposition of the lemma. Therefore Ω_F is a K-orbit of maximal dimension. $\square$

We are now in a position to prove the main theorem of the paper.

Proof of Theorem 2.1.

Firstly, we can always choose some basis $(X_1, X_2, X_3, X_4, X_5)$ of $\mathcal{G}$ such that $\mathcal{G}^1 = \mathbf{R}.X_3 \oplus \mathbf{R}.X_4 \oplus \mathbf{R}.X_5 \equiv \mathbf{R^3}$; ad_{X_1}, $ad_{X_2} \in End(\mathcal{G}^1) \equiv Mat_3(\mathbf{R})$.

It is obvious that ad_{X_1} and ad_{X_2} cannot be concurrently trivial because $\mathcal{G}^1 \cong \mathbf{R}^3$. There is no loss of generality in assuming $ad_{X_2} \neq 0$. By changing basis, if necessary, we get the similar classification of ad_{X_2} as follows:

$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $(\lambda_1, \lambda_2 \in \mathbf{R} \setminus \{1\}, \lambda_1 \neq \lambda_2 \neq 0)$; $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}$, $(\lambda \in \mathbf{R} \setminus \{0, 1\})$; $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $(\lambda \in \mathbf{R} \setminus \{1\})$; $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $(\lambda \in \mathbf{R} \setminus \{1\})$; $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}$, $(\lambda \in \mathbf{R} \setminus \{0, 1\})$; $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$; $\begin{pmatrix} cos\varphi & -sin\varphi & 0 \\ sin\varphi & cos\varphi & 0 \\ 0 & 0 & \lambda \end{pmatrix}$, $(\lambda \in \mathbf{R} \setminus \{0\}, \varphi \in (0, \pi))$.

Assume that $[X_1, X_2] = mX_3 + nX_4 + pX_5; m, n, p \in \mathbf{R}$. We can always change basis in order to have $[X_1, X_2] = mX_3$. Indeed, if

$$ad_{X_2} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, (\lambda_1, \lambda_2 \in \mathbf{R} \setminus \{1\}, \lambda_1 \neq \lambda_2 \neq 0),$$

then by changing X_1 for $X_1^{'} = X_1 + \frac{n}{\lambda_2}X_4 + pX_5$ we get $[X_1^{'}, X_2] = mX_3$, $m \in \mathbf{R}$. For the other values of ad_{X_2}, we also change basis in the same way. Hence, without restriction of generality, we can assume right from the start that $[X_1, X_2] = mX_3$, $m \in \mathbf{R}$.

There are three cases which contradict each other as follows.

(1) $[X_1, X_2] = 0$ (i.e. $m = 0$) and $ad_{X_1} = 0$. Then $\mathcal{G} = \mathcal{H} \oplus \mathbf{R}.X_1$, where $\mathcal{H}$ is the subalgebra of $\mathcal{G}$ generated by $\{X_2, X_3, X_4, X_5\}$, i.e. $\mathcal{G}$ is decomposable.

(2) $[X_1, X_2] = 0$ and $ad_{X_1} \neq 0$.

(2a) Assume $ad_{X_2} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $(\lambda_1, \lambda_2 \in \mathbf{R} \setminus \{1\},\ \lambda_1 \neq \lambda_2 \neq 0)$.

In view of Lemma 2.2, it follows by a direct computation that

$$ad_{X_1} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \xi \end{pmatrix};\ \mu, \nu, \xi \in \mathbf{R};\ \mu^2 + \nu^2 + \xi^2 \neq 0.$$

If $\xi \neq 0$, by changing $X_1^{'} = X_1 - \xi X_2$, we get

$$ad_{X_1^{'}} = \begin{pmatrix} \mu^{'} & 0 & 0 \\ 0 & \nu^{'} & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

where $\mu^{'} = \mu - \xi\lambda_1, \nu^{'} = \nu - \xi\lambda_2$. Thus, we can assume from the outset that

$$ad_{X_1} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & 0 \end{pmatrix};\ \mu, \nu \in \mathbf{R};\ \mu^2 + \nu^2 \neq 0.$$

Let $F = \alpha X_1{}^* + \beta X_2{}^* + \gamma X_3{}^* + \delta X_4{}^* + \sigma X_5{}^* \in \mathcal{G}^*$ and $U = aX_1 + bX_2 + cX_3 + dX_4 + fX_5 \in \mathcal{G}$; $\alpha, \beta, \gamma, \delta, \sigma, a, b, c, d, f \in \mathbf{R}$. So we have

$$\begin{aligned}\mathcal{G}_F &= KerB_F \\ &= \{U \in \mathcal{G} / \langle F, [U, X_i] \rangle = 0;\ i = 1, 2, , 3, 4, 5\}.\end{aligned}$$

Upon simple computation, we get

$$U \in \mathcal{G}_F \ \Leftrightarrow \ M \begin{pmatrix} a \\ b \\ c \\ d \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where

$$M := \begin{pmatrix} 0 & 0 & \mu\gamma & \nu\delta & 0 \\ 0 & 0 & -\lambda_1\gamma & -\lambda_2\delta & -\sigma \\ \mu\gamma & \lambda_1\gamma & 0 & 0 & 0 \\ \nu\delta & \lambda_2\delta & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 & 0 \end{pmatrix}.$$

Hence, $dim\Omega_F = dim\mathcal{G} - dim\mathcal{G}_F = rank(M)$. According to Lemma 2.3, Ω_F is a K-orbit of maximal dimension if $F|_{\mathcal{G}^1} \neq 0$, i.e. if $\gamma^2 + \delta^2 + \sigma^2 \neq 0$. In particular, $rank(M)$ is a constant if γ, δ, σ are not concurrently zeros. However, it is easily seen that $rank(M) = 2$ when $\gamma = \delta = 0 \neq \sigma$, but $rank(M) = 4$ when all of γ, δ, σ are different zero. This contradiction show that Case (2a) cannot happen.

(2b) In exactly the same way, but replacing the considered value of ad_{X_2} with the others, we can be seen that Case (2) cannot happen anyway.

(3) $[X_1, X_2] \neq 0$ (i.e. $m \neq 0$). By changing X_1 for ${X_1}^{'} = \frac{1}{m}X_1$ one has $[{X_1}^{'}, X_2] = X_3$. Hence, there is no loss of generality in assuming from the outset that $[X_1, X_2] = X_3$.

By an argument similar to the one in Case (2a), we get a contradiction again if $ad_{X_1} \neq 0$. In other words, $ad_{X_1} = 0$. Therefore, in the dependence on the value of ad_{X_2}, $\mathcal{G}$ will be isomorphic to one of algebras $\mathcal{G}_{5,3,1(\lambda_1,\lambda_2)}, (\lambda_1, \lambda_2 \in \mathbf{R} \setminus \{0,1\}, \lambda_1 \neq \lambda_2 \neq 0)$; $\mathcal{G}_{5,3,2(\lambda)}, (\lambda \in \mathbf{R}\setminus\{0,1\})$; $\mathcal{G}_{5,3,3(\lambda)}, (\lambda \in \mathbf{R}\setminus\{1\})$; $\mathcal{G}_{5,3,4}$; $\mathcal{G}_{5,3,5(\lambda)}$, $(\lambda \in \mathbf{R}\setminus\{1\})$; $\mathcal{G}_{5,3,6(\lambda)}$, $(\lambda \in \mathbf{R} \setminus \{0,1\})$; $\mathcal{G}_{5,3,7}$; $\mathcal{G}_{5,3,8(\lambda,\varphi)}, (\lambda \in \mathbf{R} \setminus \{0\}, \varphi \in (0,\pi)))$. Obviously, these algebras are not isomorphic to each other.

To complete the proof, it remains to show that all of these algebras are MD5-algebras. At first, we shall verify this assertion for $\mathcal{G} = \mathcal{G}_{5,3,1(\lambda_1,\lambda_2)}, (\lambda_1, \lambda_2 \in \mathbf{R}\setminus\{0,1\}, \lambda_1 \neq \lambda_2 \neq 0)$. Consider an arbitrary linear form $F = \alpha {X_1}^* + \beta {X_2}^* + \gamma {X_3}^* + \delta {X_4}^* + \sigma {X_5}^* \in \mathcal{G}^*$; ($\alpha, \beta, \gamma, \delta, \sigma \in \mathbf{R}$). We need prove that $dim\Omega_F = dim\mathcal{G} - dim\mathcal{G}_F$ is zero or maximal.

Let $U = aX_1 + bX_2 + cX_3 + dX_4 + fX_5 \in \mathcal{G}$; ($a, b, c, d, f \in \mathbf{R}$). Upon simple computation which is similar to one in Case (2a), we get

$$U \in \mathcal{G}_F \Leftrightarrow N \begin{pmatrix} a \\ b \\ c \\ d \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where

$$N := \begin{pmatrix} 0 & -\gamma & 0 & 0 & 0 \\ \gamma & 0 & -\lambda_1\gamma & -\lambda_2\delta & -\sigma \\ 0 & \lambda_1\gamma & 0 & 0 & 0 \\ 0 & \lambda_2\delta & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 & 0 \end{pmatrix}.$$

Hence, $dim\Omega_F = dim\mathcal{G} - dim\mathcal{G}_F = rank(N)$. It is plain that

$$rank(N) = \begin{cases} 0 \quad if \quad \gamma = \delta = \sigma = 0; \\ 2 \quad if \quad \gamma^2 + \delta^2 + \sigma^2 \neq 0. \end{cases}$$

Therefor, Ω_F is the orbit of dimension zero or two (maximal dimension) for any $F \in \mathcal{G}^*$, i.e. $\mathcal{G} = \mathcal{G}_{5,3,1(\lambda_1,\lambda_2)}$ is an MD5-algebra, $(\lambda_1, \lambda_2 \in \mathbf{R} \setminus \{0,1\}, \lambda_1 \neq \lambda_2 \neq 0)$. By the same way, we can be also seen that the remaining algebras are MD5-algebras. The proof is complete. $\square$

Concluding Remark

Let us recall that each real Lie algebra $\mathcal{G}$ define only one connected and simply connected Lie group G such Lie(G) = $\mathcal{G}$. Therefore we obtain a collection of eight families of connected and simply connected MD5-groups corresponding to given MD5-algebras in Theorem 2.1. For convenience, each MD5-group from this collection is also denoted by the same indices as corresponding MD5-algebra. For example, $G_{5,3,1(\lambda_1,\lambda_2)}$ is the connected and simply connected MD5-group corresponding to $\mathcal{G}_{5,3,1(\lambda_1,\lambda_2)}$. Specifically, we have eight families of MD5-groups as follows: $G_{5,3,1(\lambda_1,\lambda_2)}, (\lambda_1, \lambda_2 \in \mathbf{R}\setminus\{0,1\}, \lambda_1 \neq \lambda_2 \neq 0)$; $G_{5,3,2(\lambda)}, (\lambda \in \mathbf{R}\setminus\{0,1\})$; $G_{5,3,3(\lambda)}, (\lambda \in \mathbf{R}\setminus\{1\})$; $G_{5,3,4}$; $G_{5,3,5(\lambda)}, (\lambda \in \mathbf{R}\setminus\{1\})$; $G_{5,3,6(\lambda)}, (\lambda \in \mathbf{R}\setminus\{0,1\})$; $G_{5,3,7}$; $G_{5,3,8(\lambda,\varphi)}, (\lambda \in \mathbf{R}\setminus\{0\}, \varphi \in (0,\pi)))$. All of them are indecomposable MD5-groups. In the next paper, we shall describe the geometry of K-orbits of each considered MD5-group, topologically classify MD5-foliations associated to these MD5-groups and give a characterization of the Connes' C^*-algebras (see [1]) corresponding to these MD5-foliations.

Acknowledgement

The author would like take this opportunity to thank their teacher, Prof. DSc. Do Ngoc Diep for his excellent advice and support. Thanks are due also to the first author's colleague, Prof. Nguyen Van Sanh for his encouragement.

References

[1] **A. Connes**, *A Survey of Foliations and Operator Algebras*, Proc. Symp. Pure Math., 38(1982), 521 - 628, Part I.

[2] **Do Ngoc Diep**, *Method of Noncommutative Geometry for Group C^*-algebras*, Chapman and Hall / CRC Press Reseach Notes in Mathematics Series, #416, 1999.

[3] **A. A. Kirillov**, *Elements of the Theory of Representations*, Springer - Verlag, Berlin - Heidenberg - New York, 1976.

[4] **Vuong Manh Son et Ho Huu Viet**, *Sur La Structure Des $C^* -$ algèbres D'une Classe De Groupes De Lie*, J. Operator Theory, 11(1984), 77-90.

[5] **D. V. Tra**, *On the Lie Algebras of low dimension*, Sci. Papes of the 12th College of Institute of Math. Vietnam, Hanoi 1984 (in Vietnamese).

[6] **Le Anh Vu**, *On the Structure of the C^*-algebra of the Foliation Formed by the K-orbits of Maximal Dimension of the Real Diamond Group*, J. Operator Theory, 24(1990), 227 - 238.

[7] **Le Anh Vu**, *On the Foliations Formed by the Generic K-orbits of the MD4-Groups*, Acta Math. Vietnam, N^o 2(1990), 39 - 55.

[8] **Le Anh Vu**, *Foliations Formed by Orbits of Maximal Dimension in the Co-adjoint Representation of a Class of Solvable Lie Groups*, Vest. Moscow Uni., Math. Bulletin, Vol. 48(1993), N^o 3, 24 - 27.

[9] **Le Anh Vu**, *Foliations Formed by Orbits of Maximal Dimension of Some MD5-Groups*, East-West J. of Mathematics, Vol.5, N^o 2 (2003), 159 - 168.

[10] **Le Anh Vu and Nguyen Cong Tri**, *Some Examples on MD5 algebras and MD5 foliations Associated to Corresponding MD5 groups*, (to appear in Scientific Journal of University of Pedagogy of Ho Chi Minh city).